

# A Novel Approach to Discontinuous Bond Percolation Transition

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**Abstract.** - We introduce a bond percolation procedure on a  $D$ -dimensional lattice where two neighbouring sites are connected by  $N$  channels, each operated by valves at both ends. Out of a total of  $N$ , randomly chosen  $n$  valves are open at every site. A bond is said to connect two sites if there is at least one channel between them, which has open valves at both ends. We show analytically that in all spatial dimensions, this system undergoes a discontinuous percolation transition in the  $N \rightarrow \infty$  limit when  $\gamma = \frac{\ln n}{\ln N}$  crosses a threshold. It must be emphasized that, in contrast to the ordinary percolation models, here the transition occurs even in one dimensional systems, albeit discontinuously. We also show that a special kind of discontinuous percolation occurs only in one dimension when  $N$  depends on the system size.

Percolation transition is one of the most studied [1] critical phenomena in non-equilibrium statistical physics. It is usually modeled on a lattice [2] where bonds connecting the neighbouring sites form independently and randomly with probability  $p$ . In two and higher dimensions, a continuous phase transition to a percolating state having an infinitely large connected cluster occurs when  $p$  crosses a threshold value  $p_c > 0$  [1, 3]. One dimensional systems, on the other hand, do not show percolation transition as these systems cannot have a percolating configuration for any  $p < 1$ . The trivial fixed point  $p_c = 1$ , which can be approached only from the non-percolating region, shows signature of a continuous phase transition. In this article we propose a generic bond percolation procedure which shows a percolation transition, even in one dimension. It turns out that the transition is discontinuous.

Discontinuous percolation transition has been a subject of great interest since the study of Achlioptas *et. al.* [4]. They proposed a non-local bond percolation procedure which initially inhibits formation of a single large cluster. When applied to a fully connected graph, it apparently results in a discontinuous transition, which was named as *explosive percolation transition* (EPT). Different aspects of such explosive transition resulting from Achlioptas growth process on several other graphs have been studied in large number of following works [5, 6]. In recent studies [7] it has been shown that EPT can also be obtained from percolation rules other than Achlioptas process. In fact, it

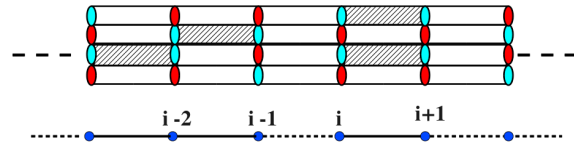


Fig. 1: (Color online) The 1D model with  $N = 4$  channels : each site has  $N = 4$  valves (discs), of which  $n = 2$  are open (blue) and the rest are closed (red). Existence of at least one channel which has open valves at both ends (shaded), represents a bond between the corresponding sites.

has been argued that certain specific rules [8] of adding new bonds, which depend on the properties of the existing clusters, are sufficient for a system to show EPT. However, discontinuous nature of EPT has been questioned recently; this originated a long debate [9] regarding the order of the transition. It has been argued that explosive percolation is, in fact, a continuous transition [10] with a small critical exponent  $\beta$  which *only* appears to be discontinuous in numerical studies. A percolation process is proposed in the following which unambiguously shows a discontinuous transition. Both the critical point and the jump in the order parameter there in the thermodynamic limit are calculated exactly.

In this Letter we introduce a local percolation procedure on a  $D$ -dimensional lattice where every pair of neighbouring sites has  $N$  different channels joining them. Each

channel contains a valve at every site which can either be open or closed independent of the other valves. Two sites are connected by a *bond* if there exists at least one channel which has open valves at both the sites. Clearly each bond appears randomly, independent of other bonds and clusters. This system shows a discontinuous transition in the thermodynamic limit when the relevant tuning parameter, namely the number of open valves  $n$  at each site, crosses a threshold value. The model could be solved exactly to locate the transition point. We show that, the transition is discontinuous in all dimensions, including 1D.

First let us describe the model in one dimension; its extension to higher dimensions is straight forward. The sites labelled by  $i = 1, \dots, (L+1)$ , are connected to their neighbours by  $N$  channels (see Fig. 1). Each channel contains a valve at every lattice site  $i$  that can either be open or closed. Out of a total of  $N$  valves at every site,  $n$  are chosen randomly and opened. The neighbouring sites of the lattice are said to be connected by a bond if there exists at least one channel between them which has open valves at both ends. The model can be recast into a simpler form by associating a set  $S_i$  of  $n$  integers, randomly chosen from a larger set  $\{1, 2, \dots, N\}$ , to each lattice site  $i$ . In this picture, a bond is said to connect two neighbouring sites  $i$  and  $j$  if  $S_i \cap S_j$  is *not null*.

This system, being one dimensional, is percolating *only* when all the  $L$  bonds are present. Obviously percolation is not possible when the number of open valves  $n$  at each site is zero. Again, the system is surely percolating for  $n > N/2$ , as in this regime every pair of sites has one or more common valves which are open. Our aim here is to find if the system is percolating for any non-zero  $n$  smaller than  $N/2$ .

The principal quantity of interest is the percolation probability  $P_L$ , which is defined as the probability that an arbitrary lattice site belongs to the spanning cluster [1, 3]. In other words,  $P_L$  is the average number of sites belonging to the spanning cluster and plays the role of the order parameter as it is nonzero *only* in the percolating regime for a thermodynamically large system. For the usual bond percolation, in two and higher dimensions,  $P_L$  vanishes continuously at the critical point for  $D \geq 2$ . It is worth mentioning that  $P_L$  should not be confused with  $\Pi_L$ , the probability that there exists a spanning cluster in the system. In fact, in the thermodynamic limit  $\Pi_\infty$  for all spatial dimensions jumps from 0 to 1 as the connection probability  $p$  is increased beyond  $p_c$ . The fact that  $\Pi_\infty$  is discontinuous, is sometimes used [1] to locate the exact transition point.

Now let us calculate  $\Pi_L$  for this multi-channel model in one dimension.

$$\Pi_L = (1 - q)^L, \quad (1)$$

where  $q$  is the probability that two neighbouring sites are not connected by a bond.  $\Pi_L$  vanishes in the thermodynamic limit  $L \rightarrow \infty$  for any  $q > 0$ , which corresponds to the connection probability  $1 - q = p < 1$ .

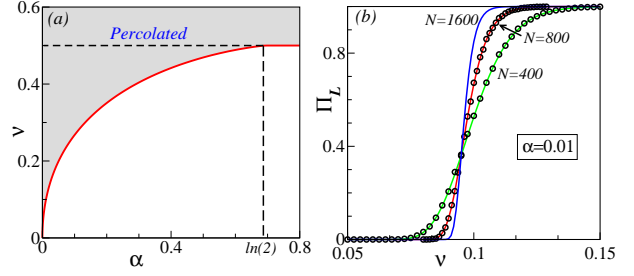


Fig. 2: (Color online) (a) Phase diagram in the  $\alpha$ - $\nu$  plane:  $\nu_c = g^{-1}(\alpha)$  is the critical line. For  $\alpha > \ln 2$  transition occurs trivially at  $\nu_c = \frac{1}{2}$ . (b) Analytical expression of  $P_L$  as a function of  $\nu$  (lines) for  $\alpha = 0.01$  and  $N = 400, 800$  and  $1600$  (correspondingly  $L = e^{\alpha N}$ ) are compared with the numerical simulations (symbols). The discontinuity, occurring at  $\nu_c = 0.095$  is more prominent for larger  $N$ .

For this model  $q$  can be calculated as follows. Since  $n$  valves can be chosen from the total  $N$  in  $C_n^N$  possible ways, the probability that  $k$  open valves are common between any two neighbouring sites is

$$Q_k = \frac{C_{n-k}^{N-n} C_k^n}{C_n^N}. \quad (2)$$

Thus, a bond between any two neighbouring sites  $i$  and  $i+1$  is absent with probability  $q = Q_0$ , which can be expressed as a function of  $N$  and  $\nu = \frac{n}{N}$  (the density of open valves at each site). Using Stirling's approximation,

$$q = \frac{(1-\nu)}{\sqrt{1-2\nu}} \exp[-g(\nu)N] \quad (3)$$

$$\text{with } g(\nu) = -\ln \left[ (1-\nu) \left( \frac{1-\nu}{1-2\nu} \right)^{(1-2\nu)} \right]. \quad (4)$$

Note that the probability  $p = 1 - q$  of connecting neighbouring sites by a bond, here, is independent of other bonds and clusters.

From Eq. (3) it is evident that  $q$  is finite, though exponentially small, for any given  $N$ . Thus, like other one dimensional models, here too, one cannot have a percolating state in the thermodynamic limit  $L \rightarrow \infty$  for any arbitrary value of  $n < N/2$ . However, for any given  $L$ , if one lets  $N \rightarrow \infty$ , then  $q$  vanishes, resulting in a percolating state for all  $n > 0$ . To explore the possibility of transition at a non-trivial  $n$ , let us couple  $N$  to the system size  $L$ . Since  $q$  is an exponentially decaying function of  $N$ , it is suggestive that one takes  $N = \mathcal{O}(\ln L)$ . Let the thermodynamic limit be taken in such a way that  $L$  approaches  $\infty$  along with  $N$ , whereas their ratio  $\alpha = \frac{\ln L}{N}$  remains fixed. From Eqs. (1) and (3), now,  $\Pi_L = (1 - cL^{-g(\nu)/\alpha})^L$  where  $c = \frac{(1-\nu)}{\sqrt{1-2\nu}}$ ; in the thermodynamic limit,

$$\lim_{L \rightarrow \infty} \Pi_L = \begin{cases} 1 & \text{for } g(\nu) > \alpha \\ e^{-c} & \text{for } g(\nu) = \alpha \\ 0 & \text{for } g(\nu) < \alpha. \end{cases} \quad (5)$$

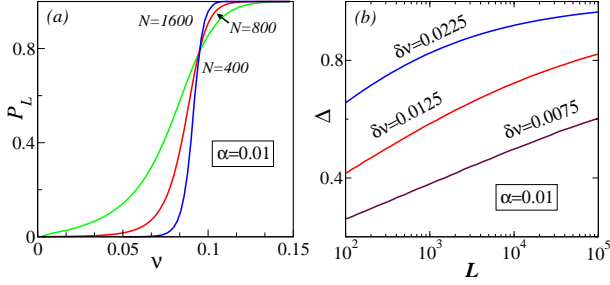


Fig. 3: (Color online) (a) The order parameter  $P_L$  as a function of  $\nu$  for  $N = 400, 800, 1600$ .  $P_L$  approaches a step function at  $\nu_c = 0.095$  as  $N \rightarrow \infty$ . (b) The jump in the order parameter,  $\Delta$  (Eq. (6)), for different values of  $\delta\nu$  approaches 1 as  $L$  is increased. In both the figures, the system size is  $L = e^{\alpha N}$  with  $\alpha = 0.01$ .

Thus, transition from a non-percolating state ( $\Pi_\infty = 0$ ) to a percolating state ( $\Pi_\infty = 1$ ) occurs at  $\nu_c = g^{-1}(\alpha)$ . The corresponding phase diagram is shown in Fig. 2(a). Note, that for  $\alpha > \ln 2$ , the transition occurs at the trivial value  $\nu_c = 1/2$ , which corresponds to half of the valves ( $n = N/2$ ) being open at every site. In Fig. 2(b) we have shown  $\Pi_L$  versus  $\nu$  for three different values of  $N$  with  $\alpha = 0.01$ . Symbols therein represent the same obtained from Monte-carlo simulations. Clearly  $\Pi_L$  approaches the step function  $\Theta(\nu - \nu_c)$  as  $L \rightarrow \infty$ .

In one dimension, the existence of a spanning cluster (i.e., when  $\Pi_\infty = 1$ ) implies that all the sites of the lattice belong to that cluster; which in turn implies that in the percolating regime, the order parameter is  $P_\infty = 1$ . This clearly indicates that the percolation transition is discontinuous. In Fig. 3(a), we have shown  $P_L$  for  $L = e^{\alpha N}$  with  $\alpha = 0.01$  and  $N = 400, 800, 1600$ . To show that  $P_L$ , like  $\Pi_L$ , also approaches  $\Theta(\nu - \nu_c)$  as  $L \rightarrow \infty$ , we estimate the jump in the order parameter  $P_L$  across the critical point  $\nu_c$  as

$$\Delta(L; \delta\nu) = P_L(\nu_c + \delta\nu) - P_L(\nu_c - \delta\nu). \quad (6)$$

In Fig. 3(b) we plot  $\Delta$  as a function of  $L$  for  $\delta\nu = 0.0075, 0.0125, 0.0225$ . The monotonic increase of  $\Delta$  with  $L$  is a clear indication that  $P_\infty$  has a discontinuity at  $\nu_c$ . For any  $\delta\nu > 0$  we have  $\lim_{L \rightarrow \infty} \Delta(L; \delta\nu) = 1$ . It is evident from the figure that this limit is approached extremely slowly.

Until now we have been discussing possibility of bond percolation transition in this one dimensional system where both  $N$  and  $L$  approach infinity keeping  $\alpha = \frac{\ln L}{N}$  fixed. If instead  $n$  is related to  $N$ , one can obtain another percolation transition. Note, that  $q$  in Eq. (3) vanishes exponentially in the limit  $N \rightarrow \infty$ , only when  $g(\nu) > 0$ . However, if  $g(\nu) \rightarrow 0$  slower than  $1/N$ ,  $q$  can approach 1 even in the  $N \rightarrow \infty$  limit. Let us discuss this scenario in details.

It is clear from Eq. (4) that  $g(\nu)$  vanishes only at  $\nu = 0$ . To the leading order in  $\nu$  we have  $g(\nu) = \nu^2$  and  $c = 1$ ,

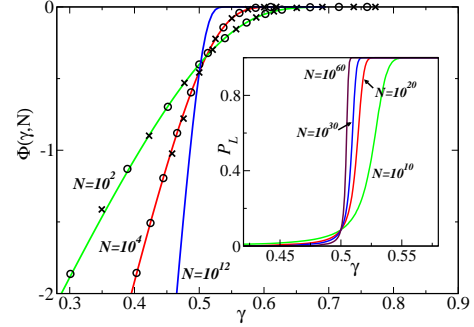


Fig. 4: (Color online)  $\Phi(\gamma, N)$  versus  $\gamma$  (main figure) and  $P_L$  versus  $\gamma$  (inset) for  $N = 10^2, 10^4$ , and  $10^{12}$ . Simulation results for  $\Phi(\gamma, N)$  with  $L = 4$  (circles) and  $L = 9$  (crosses) are also shown in the main figure. The inset shows  $P_L$  for  $L = 100$  as a function of  $\gamma$  for different  $N$ .

thus  $g(\nu)N = n^2/N$ . Let us take  $n = N^\gamma$  with  $0 < \gamma < 1$ , so that  $\nu$  approaches zero in the large  $N$  limit. This results in

$$q = 1 - p = \exp(-N^{2\gamma-1}). \quad (7)$$

Now, in the  $N \rightarrow \infty$  limit, the connection probability  $p = 1 - q$  is

$$\lim_{N \rightarrow \infty} p = \begin{cases} 0 & \text{for } \gamma < 1/2 \\ 1 - e^{-1} & \text{for } \gamma = 1/2 \\ 1 & \text{for } \gamma > 1/2. \end{cases} \quad (8)$$

Thus, again, as  $\gamma$  crosses the threshold value  $\gamma_c = \frac{1}{2}$ , i.e. when the number of open valves at each site  $n$  is increased beyond  $\sqrt{N}$ , both  $P_L$  and  $\Pi_L$  in the thermodynamic limit jump discontinuously from 0 to 1, resulting in a discontinuous percolation transition.

It is difficult to simulate this model with large  $L$  for the reason that the limiting values of  $p$  [Eq. (8)] are approached extremely slowly. Accordingly, one needs unreasonably large value of  $N$  to see the transition at  $\gamma_c$ . For example one needs  $N \sim \mathcal{O}(10^{30})$  for a system of size  $L = 100$ . This difficulty can be avoided if we measure

$$\Phi(\gamma, N) = L^{-1} \ln \Pi_L = \ln(1 - q) = \ln(1 - e^{-N^{2\gamma-1}}), \quad (9)$$

which is independent of  $L$ . A distinct signature of this discontinuous transition is that the curves  $\Phi(\gamma, N)$  versus  $\gamma$  for different values of  $N$  intersect at  $\gamma = \gamma_c$  as at this point  $\Phi(\gamma_c, N) = \ln(1 - e^{-1})$  is a constant. In Fig. 4 we have plotted  $\Phi(\gamma, N)$ , obtained numerically [symbols] for systems of size  $L = 4, 9$ , as a function of  $\gamma$  with two different  $N = 10^2, 10^4$  and compared those with Eq. (9) [solid lines]. An excellent match between these two for different  $L$ s suggests that the transition is present for systems of all sizes. It is only that the required numerical accuracy for large systems is hard to achieve in affordable computational time. A plot of Eq. (9) for  $N = 10^{12}$ , which could

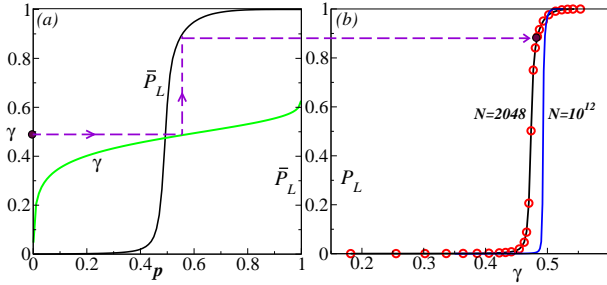


Fig. 5: (Color online) (a) The percolation probability  $\bar{P}_L$  of usual bond percolation on a  $100 \times 100$  square lattice is plotted against  $p$ , along with  $\gamma$  versus  $p$  [calculated from Eq. (7)] for our model in 2D with  $N = 2048$ . For any value of  $\gamma$ , which corresponds to a unique  $p$  (thus  $\bar{P}_L$ ) one can read out  $P_L$  using (9) [following the arrow]. The resulting  $P_L$  for  $N = 2048$  and  $10^{12}$  are shown as solid lines in (b). Symbols therein correspond to  $P_L$  obtained from direct numerical simulation of our model on a  $100 \times 100$  lattice with  $N = 2048$ .

not be supplemented with the corresponding numerical results, is shown in the same figure to demonstrate that the transition, in fact, occurs at  $\gamma_c = \frac{1}{2}$ .

Next, let us look at the order parameter  $P_L$ . Since  $p(\gamma_c) = 1 - 1/e$  for large  $N$ , one expects that the plots of  $\Pi_L$  versus  $\gamma$  for different values of  $N$  (reasonably large) would intersect at  $\gamma_c$ . This feature is clearly visible in the inset of Fig. 4, where we have plotted  $P_L$  for a system of length  $L = 100$  as a function of  $\gamma$  for  $N = 10^{10}, 10^{20}, 10^{30}$ , and  $10^{60}$ . For these large values of  $N$  we have calculated  $P_L$ , the average size of the largest cluster, by connecting the sites with probability  $p$  obtained directly from Eq. (7). It is evident that the order parameter is discontinuous in the  $N \rightarrow \infty$  limit.

Let us summarize the results obtained till now. We show that a discontinuous transition can be obtained in this one dimensional system as  $N$  and  $L$  go to infinity, in two different ways. (A) Both  $\alpha = \frac{\ln L}{N}$  and  $\nu = \frac{n}{N}$  are tuned to obtain the transition, yielding a non-trivial phase diagram (Fig. 2(a)) in the  $\alpha$ - $\nu$  plane. (B)  $\gamma = \frac{\ln n}{\ln N}$  plays the role of the control parameter resulting in a phase transition at  $\gamma_c = \frac{1}{2}$ . The natural question to ask next is, if any of these discontinuous transitions is present in higher dimensions.

The model can be extended to two dimensions in a straightforward manner; we choose to work on a square lattice. The neighbouring sites of this  $L \times L$  lattice are joined by  $N$  channels. Correspondingly, there are  $N$  valves at each site which can either be open or closed. An open valve at a site allows possibility of connection to its neighbours in *both* vertical and horizontal directions. As before, two neighbouring sites of the square lattice are connected by a bond, *only if* there exists at least one channel which has open valves at both these sites. Thus, the probability  $q$  that two neighbouring sites are not connected by a bond, when  $n$  randomly chosen valves are opened at each site, is again given by Eq. (3).

In the usual bond percolation problem [1], as mentioned earlier, the percolation transition is governed by an order parameter  $P_L$  which is defined as the probability that a randomly selected lattice site belongs to the spanning cluster. The probability that a spanning cluster exists, *i.e.*,  $\Pi_L$ , changes discontinuously across  $p_c$  in the  $L \rightarrow \infty$  limit, as in case of 1D. However, unlike one dimensional systems, the exact formula for  $\Pi_L$  is not known in higher dimensions. Henceforth we concentrate only on the order parameter, denoted by  $\bar{P}_L(p)$  for a  $L \times L$  square lattice. In the thermodynamic limit  $L \rightarrow \infty$ ,  $\bar{P}_L(p)$  vanishes continuously at the critical threshold  $p_c = 1 - q_c = \frac{1}{2}$ .

It is well known that the percolation probability  $\bar{P}_L(p)$  of a finite two dimensional system differs from  $\bar{P}_\infty(p)$  only by a correction factor which is negligibly small for large  $L$ . This indicates that a discontinuous transition similar to case (A) cannot be obtained in two dimension just by rendering  $N$  a function of  $L$ . On the other hand, for any arbitrary  $N$ , the percolation probability  $P_L$  for this model on a  $L \times L$  lattice can be obtained from  $\bar{P}_L$  as

$$P_L = \bar{P}_L(p = 1 - q), \quad (10)$$

where  $q$  is given by (3). So, the usual bond percolation that occurs as a continuous phase transition at  $p_c = 1/2 = q_c$  is also expected here for any given  $N$  when  $\nu$  increased beyond a critical value  $\nu_c$  which is a solution of  $e^{g(\nu_c)N} = 2c(\nu_c)$ . This continuous phase transition is similar to the usual bond percolation transition [3] on a square lattice.

Now we turn our attention to case (B) where  $\gamma = \frac{\ln n}{\ln N}$  is used as a tuning parameter. Here, the connection probability  $p$  has two distinct limiting values [see Eq. (8)] as  $N \rightarrow \infty$ . So, when  $\gamma$  is varied continuously in the range  $[0, 1]$  the connection probability  $p$  jumps from 0 to 1 at  $\gamma_c = \frac{1}{2}$ , resulting in a discontinuity in the percolation probability  $P_L$  across  $\gamma_c$ . Thus, this model shows a discontinuous transition also in two dimensions.

The order parameter  $P_L$  can be calculated from  $\bar{P}_L(p)$  using Eq. (10). As the analytical form of  $\bar{P}_L(p)$  is not known, we first obtain the same numerically by simulating the usual bond percolation problem on a  $L \times L$  square lattice for  $0 \leq p \leq 1$ .  $P_L$  can be read out from this data using Eq. (10) where  $p = 1 - q$  is found from Eq. (7) for any given  $\gamma$  and  $N$ . This procedure is illustrated in Fig. 5(a) for a  $100 \times 100$  lattice and  $N = 2048$ . The resulting  $P_L$  is shown in Fig. 5(b) as a solid line. The symbols therein correspond to the same obtained from direct simulation of a  $100 \times 100$  square lattice having  $N = 2048$  channels. It is clear from this figure that the transition occurs explosively near  $\gamma_c = 1/2$ .  $P_L$  for  $N = 10^{12}$ , obtained using the above procedure, is also shown in Fig. 5(b) to illustrate that, as expected, the transition point shifts towards  $\gamma_c = 1/2$  as  $N \rightarrow \infty$ .

In contrast to the usual continuous bond percolation transition, the spanning cluster appears suddenly in case of a discontinuous transition, as one approaches the transition point. In Fig. 6 we have shown the snap shots of



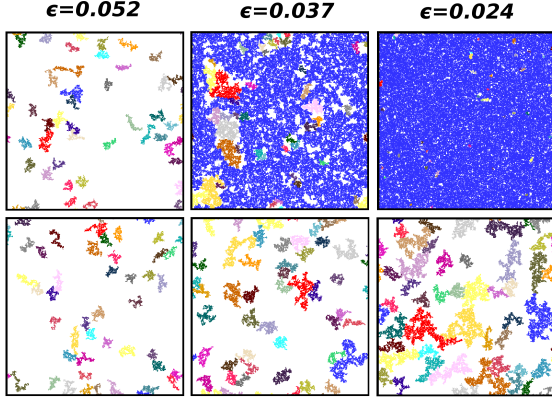


Fig. 6: (Color online) Typical configurations of this model (top panel) for  $N = 2048$  at  $\gamma = \gamma_c - \epsilon$  is compared with that of the usual bond percolation (bottom panel) at  $p = p_c - \epsilon$ . In both the cases 50 large clusters on a  $500 \times 500$  square lattice are shown.

the typical configurations (50 large clusters are shown) of our model on a  $500 \times 500$  lattice near the transition point taking  $\gamma = \gamma_c - \epsilon$  and compared the same with the usual bond percolation at  $p = p_c - \epsilon$ . Clearly, when  $\epsilon \rightarrow 0$ , the size of the largest cluster (shown as blue) grows faster in the former case, indicating the explosive nature of the transition.

It is straight forward to study these models on other kinds of lattices in two or higher dimensions. The explosive transition that occurs when the number of open channels is varied as  $n = N^\gamma$ , is quite generic. In fact the limiting values of the connection probability  $p$  [Eq.(8)] changes from being 0 to 1, at  $\gamma_c = \frac{1}{2}$  when number of channels  $N \rightarrow \infty$ . Corresponding percolation probabilities change from 0 to 1 at  $\gamma_c$  resulting in an explosive percolation in all dimensions. As discussed earlier, the other scenario where an explosive transition was obtained by tuning  $\nu = \frac{n}{N}$  and  $\alpha = \frac{\ln L}{N}$ , is specific to only one dimensional systems.

In summary, the procedure we introduce in this article, though a simple  $N$ -channel extension of the usual bond percolation, shows discontinuous percolation transition in all dimensions. Unlike the usual bond percolation transition, which is not possible in 1D and occurs as a continuous phase transition in higher dimensions, this model shows discontinuous transition in all  $D$ -dimensions, including 1D. The neighbours here are connected by  $N$  channels, each having an operating valve at either ends. Of these  $N$  valves at each site,  $n$  are open. The neighbouring sites are said to have a connecting bond if one or more channels joining them are open at both ends. We show that, when  $\gamma = \frac{\ln n}{\ln N}$  crosses a threshold value  $\gamma = \frac{1}{2}$  this system percolates abruptly in all spatial dimensions. It is rather surprising that a percolation transition occurs even in 1D at a non-trivial value of the tuning parameter. The reason lies with the fact that in the  $N \rightarrow \infty$  limit, the connection probability  $p$  changes discontinuously from

0 to 1, even though the tuning parameter  $\gamma$  is varied continuously.

Along with the above transition, another explosive percolation transition occurs especially in one dimension, driven by two parameters  $\alpha = \frac{\ln L}{N}$  and  $\nu = \frac{n}{N}$ . Occurrence of this transition owes to the fact that the percolation probability  $P_L$  in 1D explicitly depends on the system size  $L$ , unlike in higher dimensions where system size merely appears as a correction term. In fact a similar transition is possible in the usual one dimensional bond percolation scenario if one uses a  $L$ -dependent  $q = L^{-z}$ , *i.e.* the connection probability  $p = 1 - L^{-z}$ . Clearly, in the thermodynamic limit, the percolation probability  $P_L = (1 - q)^L$  jumps from 0 to 1 at  $z_c = 1$ .

We conclude with a few comments on the differences of this model with the explosive percolation. The discontinuous change in the size of spanning cluster have been reported earlier by Achlioptas *et. al.* [4] and several following works [5–7] under the name of explosive percolation transitions. Later studies, however, indicate that these transitions are in fact continuous [10], with an unusually small critical exponent. The percolation transitions reported here are analytically proven to be discontinuous. In all the studies of explosive percolation, the connection probability is allowed to evolve with the clusters, which facilitates the sudden formation of the spanning cluster resulting in an abrupt change in order parameter. In contrast, the connection probability of the model studied here does not at all depend on the existing clusters; every bond appears with the same probability. The resulting discontinuous percolation is truly an emerging behaviour.

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## REFERENCES

- [1] D. Stauffer and A. Aharony, *Introduction to Percolation Theory*, Taylor & Francis, London, 1994.
- [2] S. R. Broadbent and J. M. Hammersley, Proc. Cambridge Philos. Soc. **53**, 629 (1957).
- [3] G. Grimmett, *Percolation*, Springer, 1999.
- [4] D. Achlioptas, R. M. D'Souza, and J. Spencer, Science **323**, 1453 (2009).
- [5] Y. S. Cho, J. S. Kim, J. Park, B. Kahng, and D. Kim, Phys. Rev. Lett. **103**, 135702 (2009); F. Radicchi and S. Fortunato, Phys. Rev. Lett. **103**, 168701 (2009); H. D. Rozenfeld, L. K. Gallos, and H. A. Makse, Euro. Phys. J. **B 75**, 305 (2010).
- [6] E. J. Friedman and A. S. Landsberg, Phys. Rev. Lett. **103**, 255701 (2009); R. M. Ziff, Phys. Rev. Lett. **103**, 045701 (2009).
- [7] S. S. Manna, and A. Chatterjee, Physica **A 390**, 177 (2011); N. A. M. Araujo, and H. J. Herrmann, Phys. Rev. Lett. **105**, 035701 (2010); R. M. D'Souza, and M. Mitzenmacher, Phys. Rev. Lett. **104**, 195702 (2010).

- [8] A. A. Moreira, E. A. Oliveira, S. D. S. Reis, H. J. Herrmann, and J. S. Andrade, Phys. Rev. **E 81**, 040101R (2010).
- [9] F. Radicchi, S. Fortunato, Phys. Rev. E **81**, 036110 (2010); R. M. Ziff, Phys. Rev. E **82**, 051105 (2010); N. A. M. Araujo, J. S. Andrade Jr, R. M. Ziff, and H. J. Herrmann, Phys. Rev. Lett.**106**, 095703 (2011).
- [10] R. A. da Costa, S. N. Dorogovtsev, A. V. Goltsev, and J. F. F. Mendes, Phys. Rev. Lett.**105**, 255701 (2010); H. K. Lee, B. J. Kim, and H. Park, *arXiv:1103.4439* ; P. Grassberger, C. Christensen, G. Bizhani, S.-W. Son, and M. Paczuski, *arXiv:1103.3728*.